# Critical graphs without triangles: an optimum density construction

Wesley Pegden[*]

December 7, 2007

**Abstract**

We construct dense, triangle-free, chromatic-critical graphs of chromatic number $k$ for all $k \geq 4$. For $k \geq 6$ our constructions have $> (\frac{1}{4} - \varepsilon)n^2$ edges, which is asymptotically best possible by Turán's theorem. We also demonstrate (nonconstructively) the existence of dense pentagon-and-triangle-free $k$-critical graphs for any $k \geq 4$, again with a best possible density of $> (\frac{1}{4} - \varepsilon)n^2$ edges for $k \geq 6$. The families of triangle-free and pentagon-and-triangle-free graphs are thus rare examples where we know the correct maximal density of $k$-critical members ($k \geq 6$). We also construct dense 4-critical graphs of any odd-girth.

## 1   Introduction

The family of graphs without triangles exhibits many similarities with that of the bipartite graphs—for example, in the number of such graphs on a given number of vertices [4]—suggesting that in some senses, the most significant barrier to being bipartite is the presence of triangles. This naturally motivates the study of the chromatic number of triangle-free graphs, which, though not apparent at first sight, can be arbitrarily large, due to the constructions (for example) of Zykov and Mycielski [22], [15]. Both of these constructions have in common that they give graphs which are quite sparse. Mycielski's is the denser of the two constructions, with $O(n^{\log_2 3})$ edges, so well below quadratic edge density. Coupled with the observation that a triangle-free graph with the maximum number of edges is a balanced complete bipartite graph (and so 2-colorable), this suggests the chromatic number of triangle-free graphs with lots of edges as a natural area of study.

If we are not careful we encounter trivialities. Taking the disjoint union of a $k$-chromatic triangle-free graph with a large complete bipartite graph, we get a rather disappointing example of a $k$-chromatic triangle-free graph which is dense (in this case, it can have $\frac{1-\varepsilon}{4}n^2$ edges). Since the added complete

[*]Department of Mathematics, Rutgers University (New Brunswick), 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019. Email: pegden@math.rutgers.edu
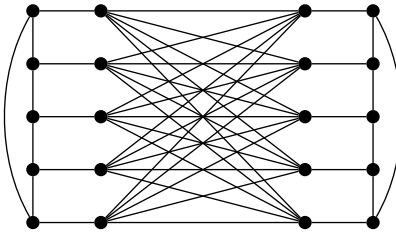
Figure 1: Toft's graph is a dense, triangle-free, 4-critical graph. It is constructed by joining 2 sets of size $2\ell + 1$ in a complete bipartite graph, and matching each to an (odd) $(2\ell + 1)$-cycle. It has $\frac{1}{16}n^2 + n$ edges.

bipartite graph plays no role in either the chromatic number of the graph or the difficulty of avoiding triangles in the 'important' part of the graph, this is hardly satisfactory.

The question along these lines by Erdős and Simonovits was: how large can the minimum degree be in a triangle-free graph of large chromatic number? Hajnal's construction given in [5] proceeds by pasting new vertices and edges onto a suitable Kneser graph (triangle-free and already with arbitrarily large chromatic number) and shows that one can have triangle-free graphs with minimum degree $\delta \geq (1 - \varepsilon)\frac{n}{3}$ and arbitrarily large chromatic number. They originally conjectured that $\delta > \frac{n}{3}$ would force 3-colorability. Häggkvist [8] found a 4-chromatic counterexample. This problem has recently been resolved: after Thomassen proved that triangle-free graphs with minimum degree $\delta > \frac{1+\varepsilon}{3}n$ have bounded chromatic number [19], Brandt and Thomassé have shown that graphs with $\delta > \frac{1}{3}n$ are indeed 4-colorable [2]. For 3-colorability, the threshold is $\frac{10}{29}$ [10]. Remaining questions appear in [2]. For more about triangle-free graphs with large minimum degree, we refer to [1].

The problem we study is of a slightly different type. We return to the basic notion of density (requiring only a quadratic number of edges, with no restrictions on the minimum degree), and use the requirement of *criticality* to avoid trivialities like adding a disjoint complete bipartite graph.

**Definition 1.1.** A graph is $k$-critical if it has chromatic number $k$, and removing any edge allows it to be properly $(k - 1)$-colored.

Thus by requiring this of our triangle-free graphs, there will be no edges contributing to the density of the graph without playing a role in its chromatic number. Many of the classical constructions of $k$-chromatic triangle-free graphs in fact give critical graphs: the Mycielski construction gives $k$-critical graphs, and the Zykov construction can be easily modified to do the same. Lower bounds on the sparsity of triangle-free critical graphs have been given by [11].

The main question we answer in the affirmative here (Theorem 1.3) is: **does there exist a family of triangle-free critical graphs of arbitrarily large chromatic number, and each with $> cn^2$ edges for some fixed $c > 0$?**
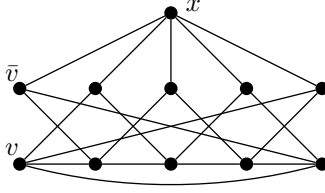
Figure 2: Mycielski's construction of $k$-critical triangle-free graphs. Shown is $\mu(C_5)$.

**Definition 1.2.** The *density constants* $c_{\ell,k}$ are each defined as the supremum of constants $c$ such that there are families of (infinitely many) $k$-critical graphs of odd-girth $> \ell$ and with $> cn^2$ edges.

(The *odd-girth* is the length of a shortest odd-cycle.) We abbreviate $c_{3,k}$ as $c_k$. Thus the positive answer to the above question will follow from our result that in fact $c_k$ is bounded below by a constant for all $k \geq 4$ (Theorem 1.3).

An easier question is to ask just for dense families of (infinitely many) triangle-free $k$-critical graphs for *each* $k \geq 4$—so, just to show that all of the constants $c_k$, $k \geq 4$ are positive. For $k = 4$, this is demonstrated by Toft's graph (Figure 1) [21], which has $> \frac{n^2}{16}$ edges (the constant $\frac{1}{16}$ is the best known for $k = 4$ even without the condition 'triangle-free'). Using Toft's graph, one can repeatedly apply Mycielski's operation [15] to get families of $k$-critical triangle-free graphs with $\geq c_k n^2$ edges for each $k \geq 4$. Here $c_k = \frac{1}{16}(\frac{3}{4})^{k-4}$, which tends to 0 with $k$—thus this method only answers the easier question. (Mycielski's operation $\mu(G)$ on $G$ is: create a new vertex $\bar{v}$ for each $v \in G$, join it to the neighbors of $v$ in $G$, and join a new vertex $x$ to all of the new vertices $\bar{v}$. This operation is illustrated in Figure 2 where $G$ is the 5-cycle.)

The weaker question is also answered directly for the particular case $k = 5$ by the following construction of Gyárfás [7]. Four copies $T_1, \ldots, T_4$ of the Toft graph are matched each with corresponding independent sets $A_1, \ldots, A_4$. The pairs $A_i, A_{i+1}$, $1 \leq i \leq 3$, are each joined in complete bipartite graphs, and the sets $A_1, A_4$ are both joined completely to another vertex $x$. This is illustrated in Figure 3. This gives a constant $c_5 \geq \frac{13}{256}$. Our construction will show that $c_5 \geq \frac{4}{31}$, which is asymptotically the same density shown by Toft [21] without the triangle-free requirement, and still the best known for that problem. Our general result for the density of $k$-critical triangle-free graphs is the following, which we prove constructively.

**Theorem 1.3.** *There is a dense family of triangle-free critical graphs containing infinitely many $k$-chromatic graphs for each $k \geq 4$. In particular, the density constants satisfy $c_4 \geq \frac{1}{16}$, $c_5 \geq \frac{4}{31}$, and $c_k = \frac{1}{4}$ for all $k \geq 6$.*

In the case $k = 4$ our construction is the same as Toft's graph. For $k \geq 6$, the upper bound of $\frac{1}{4}$ follows from Turán's theorem just from the fact that the graphs don't have triangles.

Theorem 1.3 follows from our constructions in Section 2, which can be seen as generalizing the Toft graph. The basic idea of the construction is the following:
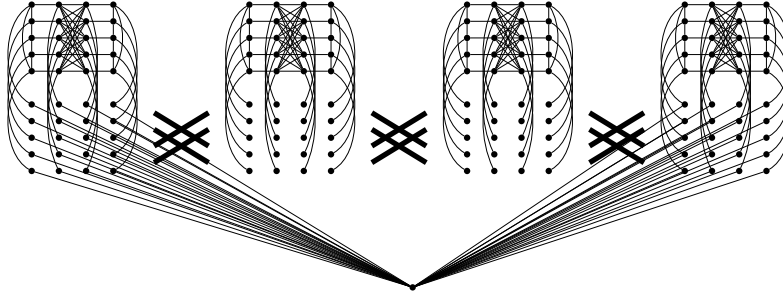
3

Figure 3: Gyárfás's graph: a dense, triangle-free, 5-critical graph. Each of the three double crosses stands for the edges of the complete bipartite graph joining the independent sets on either side. (For this particular $n = 161$, each double cross represents 400 edges). Gyárfás's graph is dense, with $(\frac{13}{256} + o(1))n^2$ edges.

the Toft and Gyárfás constructions work by pasting graphs onto independent sets so that in any $(k-1)$-coloring of the graph, at least two different colors occur in each of the independent sets (and the 'pasting' has suitable criticality-like properties as well). In Section 2, we recursively define a way of 'critically' pasting graphs to independent sets so that we can be assured that in a $(k-1)$-coloring of a graph, $\lfloor \frac{k}{2} \rfloor$ colors occur in each independent set (i.e., $\lfloor \frac{k}{2} \rfloor$ in one and $\lceil \frac{k}{2} \rceil$ in the other). Moreover, the independent sets will be nearly as large as the new 'pasted' parts. From this point, we simply defer back to the idea of Toft's construction, joining two such independent sets in a complete bipartite graph. (One could also use Gyárfás's graph as model for the general case, but this gives less dense graphs.)

A natural extension of the triangle-free case is of course to avoid higher-order odd-cycles. Our result for avoiding pentagons (as well as triangles) is the following:

**Theorem 1.4.** *There is a dense family of pentagon-and-triangle-free critical graphs whose members have unbounded chromatic number. In particular, the density constants satisfy $c_{5,4} \geq \frac{1}{36}$, $c_{5,5} \geq \frac{3}{35}$, and $c_{5,k} = \frac{1}{4}$ for $k \geq 6$.*

Theorem 1.4 is nonconstructive, since one step of the 'construction' involves a deletion argument. The proof is in Section 3, and combines the idea of our construction for Theorem 1.3 with a modified Mycielski operation. The upper bound of $c_{5,k} \leq \frac{1}{4}$ is a consequence of the classical Erdős-Stone theorem [6], since odd-cycles are 3-chromatic.

Theorems 1.3 and 1.4 determine the exact values of $c_{3,k}$ and $c_{5,k}$, respectively, for $k \geq 6$ and so the triangle-free and pentagon-and-triangle-free families are perhaps the only examples of natural families of graphs where the maximum possible edge-density in $k$-critical graphs is now asymptotically known ($k \geq 6$); moreover, in this case, the requirement of $k$-criticality does not affect the maximum possible edge density (asymptotically).

For higher-order odd cycles ($\ell \geq 7$), we have a quadratic density result only for $k = 4$:

4

**Theorem 1.5.** *For each $\ell \geq 3$, there is a dense family of (infinitely many) 4-critical graphs of odd-girth $> \ell$. In particular,*

$$c_{\ell,4} \geq \frac{1}{(\ell+1)^2}. \tag{1}$$

Theorem 1.5 follows from our construction in Section 4 (which uses an extension of the Mycielski operation). In terms of the constants $c_{\ell,k}$, the results of Theorems 1.3 through 1.5 are summarized in Table 1 (in Section 5).

## Acknowledgment

I'd like to thank András Gyárfás for some helpful discussions on this question; particularly, for giving his construction (Figure 3) for the case $k = 5$.

## 2 Avoiding triangles (constructions)

To construct critical triangle-free graphs, we will construct graphs $U$ which have large independent sets (essentially as large as $U$ for $k \geq 6$ in fact) in which $\left\lfloor \frac{k}{2} \right\rfloor$ colors must show up in a $(k-1)$-coloring. (In fact, our techniques would allow us to require that all $k - 1$ colors show up in the independent set, but this is not the best choice from a density perspective.) The $U$'s will have criticality-like properties as well (Lemma 2.5), and we will be able to join the relevant independent sets from two such $U$'s in a complete bipartite graph to get our construction. Let us now specify the construction of the sets $U$.

Given graphs $S_1, S_2, \ldots S_t$, we construct $U(S_1, S_2, \ldots, S_t)$ as follows: take the (disjoint) union of the graphs $S_i$, together with the independent set $A = \prod V(S_i)$, which we call the *active* set of vertices, and join vertices $v \in S_i$ to vertices $u \in A$ which equal $v$ in their $i$'th coordinate. This construction is illustrated for $t = 2$ in Figure 4. Where $r_1, r_2, \ldots r_t$ are positive integers, we write $U(r_1, r_2, \ldots r_t)$ to mean a construction $U(S_1, S_2, \ldots, S_t)$ where, for each $i$, $S_i$ is a triangle-free $r_i$-critical graph without isolated vertices.

With just one graph $S$, $U(S)$ simply consists of a matching between $S$ and an independent set of the same size. Thus the Toft graph is two identical copies of a $U(3)$, whose active sets are joined in a complete bipartite graph. Gyárfás's construction is now four copies $C_1, C_2, C_3, C_4$ of a $U(4)$, where the active sets of $C_i$ and $C_{i+1}$ are joined in a complete bipartite graph for $1 \leq i \leq 3$, and the active sets of both $C_1$ and $C_4$ are completely joined to a new vertex $x$.

**Observation 2.1.** *$U(r_1, r_2, \ldots, r_t)$ is triangle-free, and if $s_i = |S_i|$, it has $s_1 \cdots s_t$ active vertices, and $s_i$ structural vertices of type $i$.* □

The *structural vertices* of type $i$ are vertices from the copies of $S_i$ used in the construction of $U(r_1, r_2, \ldots r_t)$.

Our inductive proof of Lemma 2.5 will use the following properties of the sets $U(S_1, \ldots, S_t)$:
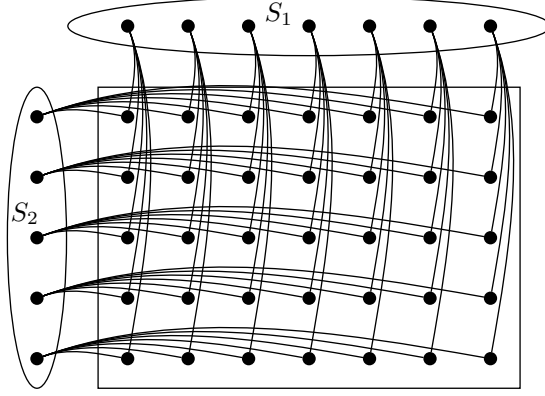
5

Figure 4: Constructing $U(S_1, S_2)$ from the graphs $S_1$ and $S_2$. The rectangle encloses the set of active vertices. (Edges within the $S_i$ are not drawn.)

**Observation 2.2.** *Let $A_\omega \subset \prod S_i$ be the all the vertices of the active set whose $t$th coordinate is the vertex $\omega \in S_t$. Then the subgraph of $U(S_1, \ldots S_t)$ induced by the set $A_\omega \cup \bigcup_{1 \leq i < t} V(S_i)$ is isomorphic to $U(S_1, \ldots, S_{t-1})$.* ☐

**Observation 2.3.** *Any colorings of the graphs $S_1, S_2, \ldots S_t$ can be extended to the rest of $U(S_1, S_2, \ldots, S_t)$ (i.e., to the active vertices) so that just $t + 1$ different colors of our choosing appear in the active set.*

*Proof.* This follows from observing that the degree of each active vertex in $U(S_1, S_2, \ldots, S_t)$ is $t$. ☐

Before proceeding any further, we point out the following basic fact, which lies at the heart of all recursive constructions of critical graphs:

**Observation 2.4.** *A $k$-critical graph without isolated vertices can be $k$-colored so that color $k$ occurs only at a single vertex of our choice.*

*Proof.* This follows from observing that such a graph is *vertex-critical*; i.e., removing any vertex decreases the chromatic number. ☐

We are now ready to prove the main lemma, about the sets

$$U_{k-i+1}^{k-1} \stackrel{\text{defn}}{=} U(k-1, k-2, \ldots, k-i+1).$$

Basically, it says that in a $(k-1)$-coloring, their active sets have a set of properties similar to criticality.

**Lemma 2.5.** *The sets $U_{k-i+1}^{k-1} = U(k-1, k-2, \ldots, k-i+1)$ satisfy:*

1. *$U_{k-i+1}^{k-1}$ can be properly $(k-1)$-colored so that $\leq i$ different colors appear at active vertices and so that among active vertices, only one vertex of our choosing gets the $i$th color.*

6

2. *In any $(k-1)$-coloring of $U_{k-i+1}^{k-1}$, at least $i$ different colors occur as the colors of active vertices.*

3. *If any edge from $U_{k-i+1}^{k-1}$ is removed, it can be properly $(k-1)$-colored so that at most $i-1$ colors occur at active vertices.*

*Proof.* We prove Lemma 2.5 by induction on $i$. For $i = 1$, we interpret $U_k^{k-1}$ as simply a single active vertex and the statement is trivial. Recall that the graphs $S_j$ $(1 \le j \le i-1)$ are the triangle-free $(k-j)$-critical graphs used in the construction of $U_{k-i+1}^{k-1}$. For $i > 1$, Observation 2.2 tells us that $U_{k-i+1}^{k-1}$ consists of $s_{i-1} = |S_{i-1}|$ copies $C_\omega$ of a $U_{k-i+2}^{k-1}$, one for each vertex $\omega \in S_{i-1}$, which are pairwise disjoint except at the graphs $S_1, S_2, \ldots, S_{i-2}$ (in other words, their active sets $A(C_\omega)$ are disjoint). Note that each $\omega \in S_{i-1}$ is adjacent to every vertex in $A(C\omega)$.

To prove part 1, color $S_{i-1}$ with colors $i-1$ through $k-1$ so that color $i-1$ occurs only at vertex $\omega_0$ (we can do this by Observation 2.4). By induction, $(k-1)$-color $C_{\omega_0}$ so that its active vertices get colors from $1, 2, \ldots, i-2, i$ and so that color $i$ occurs at only one vertex $u$ of $C_{\omega_0}$. Now we can use Observation 2.3 to extend the current partial $(k-1)$-coloring to the rest of the $C_\omega$'s so that their active vertices get colors from 1 through $i-1$; this gives us a $(k-1)$-coloring of $U_{k-i+1}^{k-1}$ where active vertices get colors from 1 through $i$, and $i$ occurs only at $u$. Finally, note that we could have chosen so that $u$ was any active vertex.

For part 2, notice that by induction, $i-1$ different colors occur at active vertices of the $C_\omega$'s in a $(k-1)$-coloring of $U_{k-i+1}^{k-1}$. Thus if only $i-1$ colors appear at active vertices of $U_{k-i+1}^{k-1}$, the same set of $i-1$ colors occurs at the sets of active vertices of each of the $C_\omega$'s; but then $S_i$ is colored with a disjoint set of colors, but then we need $(i-1) + (k-i+1) > k-1$ colors overall, a contradiction.

For part 3, assume first that the removed edge came from $S_i$: Then we can $(k-i)$-color what remains, and (inductively by part 1) color the $C_\omega$'s so that only the leftover $i-1$ colors occur at the active vertices.

If the removed edge was one joining a vertex $\omega_0$ in $S_i$ to a vertex $v$ in the active set of $C_{\omega_0}$, we color $S_i$ with the colors $i-1$ through $k-1$ so that color $i-1$ occurs only at vertex $\omega_0$ (by Observation 2.4), and color $C_{\omega_0}$ so that only colors $1, 2, \ldots, i-1$ occur at its active vertices, and so that color $i-1$ occurs only at $v$ among active vertices (by part 1, inductively). Coloring the other $C_\omega$'s so that only colors $1, 2, \ldots, i-1$ occur at their active vertices (again part 1), we have a proper $(k-1)$-coloring of $U_{k-i+1}^{k-1}$ where only colors 1 through $i-1$ appear at active vertices.

If instead the removed edge was from one of the copies $C_{\omega_0}$, we color that copy (by induction) so that only colors $1, 2, \ldots, i-2$ occur at its active vertices, and $S_i$ with colors $i-1$ through $k-1$ so that $i-1$ occurs only at the vertex $\omega_0$ (by Observation 2.4). Then coloring the rest of the $C_\omega$ so that only colors 1 through $i-1$ appear at active vertices (by part 1), we have a proper $(k-1)$-coloring of $U_{k-i+1}^{k-1}$ where only colors 1 through $i-1$ appear at active vertices. $\square$

## 2.1 Constructing $G_k$

We take $G_k$ to be the union of a $C_1 = U^{k-1}_{\lceil \frac{k}{2} \rceil +1}$ with a $C_2 = U^{k-1}_{\lfloor \frac{k}{2} \rfloor +1}$; the active sets are joined in a complete bipartite graph. (For $k = 4$ this gives the Toft graph.) $G_k$ is $k$-colorable, since part 1 of Lemma 2.5 implies that we can $(k-1)$-color each $C_i$ so that the sets of active vertices of each get just $\lfloor \frac{k}{2} \rfloor$ and $\lceil \frac{k}{2} \rceil$ colors, respectively. Moreover part 2 of Lemma 2.5 implies that $G_k$ cannot be $(k-1)$-colored. $G_k$ is triangle-free since the $C_i$'s are triangle-free, the new edges form a bipartite graph, and they are added to an independent set of vertices.

For criticality there are now just two cases to check. If a removed edge belongs to (for example) $C_1$, then by part 3 of Lemma 2.5 we can color $C_1$ with colors $1, \ldots, k-1$ so that only colors $1, \ldots, \lfloor \frac{k}{2} \rfloor - 1$ occur at its active vertices. Then, coloring $C_2$ with $1, \ldots, k-1$ so that only $\lfloor \frac{k}{2} \rfloor, \lfloor \frac{k}{2} \rfloor + 1, \ldots, k-1$ occur at its active vertices, we have given a proper $(k-1)$-coloring.

Otherwise the removed edge is between active vertices of $C_1$, and $C_2$. We can color each of the $C_i$ with colors $1, \ldots, k-1$ so that only colors $1, \ldots, \lfloor \frac{k}{2} \rfloor$ appear at active vertices of $C_1$, and only $\lfloor \frac{k}{2} \rfloor, \ldots, k-1$ appear at active vertices of $C_2$, and (by part 1 of Lemma 2.5) require that color $\lfloor \frac{k}{2} \rfloor$ occurs only at the end-vertices of the removed edge, so that this $(k-1)$-coloring is proper.

## 2.2 Density of $G_k$ (for $k \geq 6$)

To estimate the density of $G_k$, recall that $C_p$, $p \in \{1, 2\}$, will have $s^p_1 s^p_2 \cdots s^p_{t_p}$ active vertices and $s^p_j$ structural vertices of type $j$, where here $s^p_j$ is the number of vertices in the graph $S^p_j$, the $(k-j)$-critical graph used in the construction of $C_p = U^{k-1}_{\lfloor \frac{k}{2} \rfloor +1} = U(k-1, k-2, \ldots, \lfloor \frac{k}{2} \rfloor + 1)$.

Now so long as $k \geq 6$, we have that $t_1, t_2 \geq 2$ and so by choosing $s^p_1, s^p_2$ suitably large for each $p \in \{1, 2\}$, we assure that a $1 - \varepsilon'$ fraction of the vertices from each $C_p$ are active, giving that $G_k$ has $\geq (\frac{1}{4} - \varepsilon)n^2$ edges for all $k \geq 6$, so long as we can take $C_1, C_2$ to be the same size.

For the case where $k$ is even, we can let them be identical, so if we are not interested in good constants for the case where $k$ is odd, we have finished the proof of Theorem 1.3, since one can use the Mycielski operation to get the case '$k$ is odd' from the case '$k$ is even' (giving a density constant of $\frac{3}{16} - \varepsilon$ for the odd $k$). To prove better density constants, we will want to argue that we can take $C_1, C_2$ to be the same size even if $k$ is odd. This will follow from Lemma 2.8 below.

An arithmetic progression is called homogeneous if it includes the point 0. We call a subset of $\mathbb{Z}^+$ *positive homogeneous* if it contains (all) the positive terms of an infinite homogeneous arithmetic progression. We have:

**Observation 2.6.** *The sums and products of positive homogeneous sets are positive homogeneous sets.* □

The important property we need is this:

**Observation 2.7.** *The intersection of two positive homogeneous sets is a positive homogeneous set.* □

In particular, the intersection is infinite, and so nonempty. Because of this property, the following lemma regarding the $U_{k-i+1}^{k-1}$ implies that we can let $|C_1| = |C_2|$ in the case where $k$ is odd, finishing the proof of the density of the $G_k$:

**Lemma 2.8.** *Let $k \geq 4$, $1 \leq i \leq k$, and fix triangle-free $(k-j)$-critical graphs $S_j$ of order $s_j$ for each $j > 2$. Then the set of natural numbers $n$ for which there is a $U_{k-i+1}^{k-1}$ on $n$ points constructed using the fixed graphs $S_2, S_3, \dots$ (so, we only have freedom in choosing the graph $S_1$) is positive homogeneous. The same holds for the set of natural numbers $n'$ for which there exists a $U_{k-i+1}^{k-1}$ constructed from the $S_2, S_3, \dots$ which has exactly $n'$ active vertices.*

*Proof.* This follows by induction from Observations 2.1, 2.6, and 2.7. □

This completes the proof that the $G_k$ are dense with $(\frac{1}{4} - o(1))n^2$ edges for $k \geq 6$. For $k = 4$ our construction is the same as Toft's graph and so has $\frac{1}{16}n^2 + n$ edges. The remaining case is $k = 5$.

## 2.3 Density of $G_5$

In this section we show the bound $c_5 \geq \frac{4}{31}$.

In the construction of $G_5$, $C_1$ is a $U(4)$, and $C_2$ is a $U(4,3)$. Say $C_1$ consists of a copy $S_1^1$ of the Toft graph matched with an independent set of the same size, and $C_2$ is likewise constructed from the 4- and 3-critical graphs $S_1^2, S_2^2$, respectively. Denoting by $v(H)$ and $e(H)$ the number of vertices and edges, respectively, in a graph $H$, we have:

$$e(G_5) > e(S_1^1) + v(S_1^1) \cdot |A(C_2)|.$$

(Of course, here $|A(C_2)| = v(S_1^2) \cdot v(S_2^2)$.) Since $S_1^1$ is the Toft graph, we have

$$e(G_5) > \frac{1}{16} v(S_1^1)^2 + v(S_1^1) \cdot |A(C_2)|. \tag{2}$$

For vertices instead of edges, we have

$$v(G_5) = 2v(S_1^1) + v(S_1^2) \cdot v(S_2^2) + v(S_1^2) + v(S_2^2).$$

When we take $v(S_1^2)$, $v(S_2^2)$ to be large, the last two summands are negligible:

$$v(G_5) < (1 + \varepsilon) \left( 2v(S_1^1) + |A(C_2)| \right). \tag{3}$$

We want to maximize the ratio $\frac{e(G_5)}{v(G_5)^2}$. It is an easy optimization problem to check that our best choice is to make $|A(C_2)|$ larger than $v(S_1)$ by a factor of $\frac{15}{8}$ (note that we can do this by Lemma 2.8 and Observation 2.7). For this choice, and denoting now $v(S_1^1)$ by simply $s_1^1$, lines (2) and (3) give:

$$\frac{e(G_5)}{v(G_5)^2} > \frac{1}{1 + \varepsilon} \frac{\frac{1}{16}(s_1^1)^2 + \frac{15}{8}(s_1^1)^2}{(\frac{31}{8})^2 (s_1^1)^2} = \left( \frac{1}{1 + \varepsilon} \right) \frac{4}{31}, \tag{4}$$

and thus, taking the supremum (as $n$ grows large), we do have that $c_5 \geq \frac{4}{31}$.

# 3  Avoiding pentagons (existence result)

By using a partial Mycielski operation together with the constructive method used to prove Theorem 1.3 in Section 2, we will demonstrate the existence of dense $k$-critical graphs for $k \geq 4$ whose shortest odd-cycles have length 7. In this section we prove the bounds $c_{5,4} \geq \frac{1}{36}$, $c_{5,5} \geq \frac{3}{35}$, and $c_{5,k} \geq \frac{1}{4}$ for $k \geq 6$ (thus $c_{5,k} = \frac{1}{4}$ for $k \geq 6$ by the Erdős-Stone theorem).

Beginning with a $(k-1)$-critical pentagon-and-triangle-free graph $M_{k-1}$, double the vertex set by adding new vertices $\bar{v}$ for each $v$, and joining $\bar{v}$ to all the vertices in the neighborhood $\Gamma(v) \subset M_{k-1}$ to make a graph $\bar{M}_{k-1}$. (If we now added another vertex $x$ and joined it to to all the new vertices $\bar{v}$, this would be the Mycielski operation illustrated in Figure 2.) We say the original edges of $M_{k-1}$ are the Type 1 edges of $\bar{M}_{k-1}$, the newly added edges are the Type 2 edges. We call the added vertices $\bar{v}$ the *forward* vertices of $\bar{M}_{k-1}$ (notice that they form an independent set). One can check that $\bar{M}_{k-1}$ is $(k-1)$-colorable; that in any $(k-1)$-coloring of $\bar{M}_{k-1}$, $k-1$ colors appear at forward vertices; and moreover, that if one removes any edge from $\bar{M}_{k-1}$, it can be $(k-1)$-colored so that $\leq k-2$ colors appear at the set of forward vertices. With respect to odd cycles, the important aspect of the doubling operation is the following:

**Lemma 3.1.** *If $M$ is a graph of odd-girth $\ell + 2$, then $\bar{M}$ has odd-girth $\ell + 2$.*

*Proof.* Every vertex in $\bar{M}$ is either a vertex $v \in M$ or its corresponding $\bar{v}$. If we 'project' all of the vertices $\bar{v}$ onto their corresponding vertices $v$, this induces a map from the edges of $\bar{M}$ to the edges of $M$. This map sends walks to walks and closed walks to closed walks (of the same lengths). Thus if $\bar{M}$ contains some odd cycle of length $\leq \ell$, $M$ contains an odd closed walk of length $\leq \ell$, and thus an odd cycle of length $\leq \ell$. $\qquad\square$

Now to construct graphs $M_{k-1}^r$ from $\bar{M}_{k-1}$, we remove Type 2 edges one by one, discarding any forward vertices isolated by this process, until we cannot do so without allowing the result to be $(k-1)$-colored so that only $r-1$ colors appear at forward vertices. (This is the step where our argument is nonconstructive.) The result satisfies:

**Observation 3.2.** *$M_{k-1}^r \subset \bar{M}_{k-1}$ is a (connected) subgraph such that*

1. *$M_{k-1}^r$ can be $(k-1)$-colored so that just $r$ colors appear at forward vertices. Moreover, one of those $r$ colors can be required to appear (among forward vertices) at only a single vertex of our choosing.*

2. *In any $(k-1)$-coloring of $M_{k-1}^r$, at least $r$ colors appear at forward vertices,*

3. *If any edge is removed from $M_{k-1}^r$, the result can be $(k-1)$-colored so that at most $r-1$ colors appear at forward vertices.* $\qquad\square$

In spite of the fact that these are essentially the same properties enumerated for the sets $U_{k-i+1}^{k-1}$ in Lemma 2.5 (with the terminology *active vertices* replaced

with *forward vertices*), the graphs $M_{k-1}^r$ will not take their place in our constructions in this section. Instead, we will use the $M_{k-1}^r$ (actually, their forward vertices) in place of the graphs $S_{k-r}$ in the construction of the analogue to the $U_{k-i+1}^{k-1}$ used here. Let us now make this precise.

Fixing the parameter $k$, we construct graphs $W(r_1, r_2, \ldots, r_t)$ as follows: take a disjoint union of graphs $M_{k-1}^{r_i}$, $1 \le i \le t$, together with an 'active set' $A = \prod F(M_{k-1}^{r_i})$ (here $F(M)$ denotes the forward vertices of $M$); and join, for each $i$, each vertex $\bar{v}$ in $F(M_{k-1}^{r_i})$ to all the vertices $u \in A$ which equal $\bar{v}$ in their $i$th coordinate. Similar to our convention in Section 2, we set

$$W_{k-i+1}^{k-1} \overset{\text{defn}}{=} W(k-1, k-2, \ldots, k-i+1).$$

The next lemma is the analogue to Lemma 2.5 for the sets $W_{k-i+1}^{k-1}$.

**Lemma 3.3.** *The sets $W_{k-i+1}^{k-1} = W(k-1, k-2, \ldots, k-i+1)$ satisfy:*

1. *$W_{k-i+1}^{k-1}$ can be properly $(k-1)$-colored so that $\le i$ different colors appear at active vertices and so that among active vertices, only one vertex of our choosing gets the $i$th color.*

2. *In any $(k-1)$-coloring of $W_{k-i+1}^{k-1}$, at least $i$ different colors occur as the colors of active vertices.*

3. *If any edge from $W_{k-i+1}^{k-1}$ is removed, it can be properly $(k-1)$-colored so that at most $i-1$ colors occur at active vertices.*

*Proof.* The proof of Lemma 3.3 is essentially identical to that of Lemma 2.5. It should satisfy the reader to check that the properties of the forward sets of the $M_{k-1}^r$ under $(k-1)$-colorings enumerated in Observation 3.2 are shared by the $r$-critical graphs $S_{k-r}$, and are, moreover, precisely those properties of the $S_{k-r}$ used in the proof of Lemma 2.5. $\square$

Similar to our construction in Section 2.1, we take $G_k^5$ to be the union of a $C_1 = W_{\lceil \frac{k}{2} \rceil + 1}^{k-1}$ with a $C_2 = W_{\lfloor \frac{k}{2} \rfloor + 1}^{k-1}$; the active sets are joined in a complete bipartite graph. As in that section, we have (now as consequences of 3.3) that $G_k^5$ is critically $k$-colorable.

We now check that $G_k^5$ contains no odd cycles of length $\le 5$. If we were delete from $G_k^5$ all the subgraphs $M_{k-1} \subset M_{k-1}^r$ ($M_{k-1}$ is the original $(k-1)$-critical graph before the doubling and deletion operations), we are left with a bipartite graph (note, for example, that if in the result we contract each of the independent sets $A(C_1)$, $A(C_2)$ to a point, what is left is a tree, which can therefore contain no closed walks of odd length). Thus any odd cycle in $G_k^5$ must contain a vertex $v$ from one of the original $M_{k-1}$ graphs—without loss of generality we let the vertex $v$ be from $C_1$. By Lemma 3.1, a triangle or pentagon in $G_k^5$ cannot lie entirely in one of the graphs $M_{k-1}^r$, thus it must include at least some active vertex in $C_1$. Can it contain just one active vertex? No: if we remove all active vertices but one from $C_1$, the result is some copies of some $M_{k-1}^r$'s (where $r$ takes some values $\le k-1$) which will be disconnected by

11

the removal of the remaining active vertex. Thus the result contains no cycles passing through the active vertex, and we see that any cycle in $C_1$ passing through one active vertex must pass through at least two. Of course, since removing the active vertices of $C_1$ from $G_k^5$ separates what is left of $C_1$ from $C_2$, it is true more generally that any cycle in $G_k^5$ passing through an active vertex of $C_1$ must in fact pass through two such vertices. The distance between two such vertices is 2, and the distance from each to $v$ is at least 2, so the length of any odd cycle in $G_k^5$ is at least 7.

Finally, we remark on the density of the graphs $G_k^5$. From a density standpoint, it seems perhaps that there is a problem with our use of the graphs $M_{k-1}^r$ in the construction of the graphs $W_{k-i+1}^{k-1}$; we have not given any argument that restricts how small the forward-sets will be of the $M_{k-1}^r$ relative to all of the $M_{k-1}^r$, so how will we argue that the active sets of the $W_{k-i+1}^{k-1}$ ($i \geq 3$) can take up a $(1-\varepsilon)$ fraction of those graphs? The construction of each $W_{k-i+1}^{k-1}$ involves the use of a $M_{k-1}^{k-1}$, and this graph does not require a deletion argument—we know that half of $M_{k-1}^{k-1}$ consists of forward vertices. It is not hard to see then, that it is sufficient for our purposes to know that there are graphs $M_{k-1}^{k-2}$ for which the forward sets $F(M_{k-1}^{k-2})$ are arbitrarily large (even if $|F(M_{k-1}^{k-2})|$ is small compared with $|M_{k-1}^{k-2}|$). The following simple observation will suffice for our purposes:

**Observation 3.4.** *The forward set of an $M_{k-1}^r$, $2 \leq r \leq k-1$, must dominate the starting graph: we have that every vertex in $M_{k-1} \subset M_{k-1}^r$ is adjacent to some vertex in the forward set of $M_{k-1}^r$.*

*Proof.* Otherwise, $M_{k-1}^r$ can be $(k-1)$-colored so that just *one* color appears at the forward vertices. $\square$

It is now straightforward to check that, for example, the $(k-1)$-critical triangle-and-pentagon-free graphs $G_{k-1}^5$ given by our recursion cannot be dominated by sets of bounded size, and so we can, as desired, construct graphs $M_{k-1}^{k-2}$ whose forward sets are arbitrarily large, and thus can construct graphs $W_{k-i+1}^{k-1}$ in which the active set occupies a $(1-\varepsilon)$ fraction of the vertex set.

It is easy to check that our construction gives $c_{5,4} \geq \frac{1}{36}$, and, with the remarks from the preceding paragraphs out of the way, similar methods as those in Section 2.2 give that $c_{5,k} = \frac{1}{4}$ for all $k \geq 6$. The bound $c_{5,5} \geq \frac{3}{35}$ follows from an argument (which we omit) similar to that in Section 2.3; the best choice here is to let the larger of the two active sets be bigger by a factor of $\frac{17}{6}$.

# 4    Avoiding more odd cycles ($\ell \geq 7$) for $k = 4$

For the particular case of $k = 4$, we will give a construction of dense, 4-critical graphs of arbitrarily large odd-girth. For this we use an extension of the Mycielskian doubling operation: to construct $\mu^q(M)$, add for each vertex $v = v^0$ in the graph $M$ vertices $v^1, v^2, \ldots, v^q$ and join $v^i$ to all vertices $w^{i-1}$ (and $w^{i+1}$)
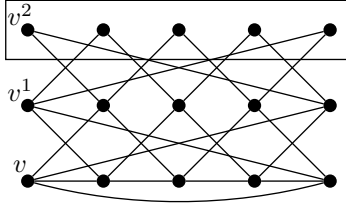
Figure 5: An extension of the Mycielskian operation: shown is $\mu^2(C_5)$. The box encloses 'active' vertices.

for which $v, w$ are adjacent in $M$. This is demonstrated in Figure 5 with the pentagon for $q = 2$. We will call all the vertices $v^q$ ($v \in M$) the active vertices. We have an extension of Lemma 3.1:

**Lemma 4.1.** *If $M$ has odd-girth $\ell + 2$, then $\mu^q(M)$ has odd-girth $\ell + 2$.*

*Proof.* As in the proof of Lemma 3.1, we can 'project' all the vertices $v^i$ onto their corresponding $v^0$ to give a function from $\mu^q(M)$ to $M$ which preserves walks and closed walks. Thus if $\mu^q(M)$ contains an odd-cycle of length $\leq \ell$, so does $M$. $\qquad\square$

If we now were to join a new vertex $x$ to the active vertices of $\mu^q(M)$, we would have what has been called a *cone* over the graph $M$ [18]. To construct dense, critical graphs with large odd-girth using the $\mu^q(M)$, we would like to claim that if we start with a $(k-1)$-critical graph $G$, then in a $(k-1)$-coloring of $\mu^q(M)$, all $(k-1)$ colors would have to appear in the active set. From here we could use suitable deletion arguments and proceed as in previous sections to extend Theorem 1.4 to arbitrarily large odd girth (albeit with worse constants as the odd-girth increases).

For $q > 1$, however, it is not true in general that for a $(k-1)$-critical graph $M$, all $k-1$ colors show up at active vertices in a $(k-1)$ coloring of $\mu^q(M)$. In the special case where $k = 4$ and so $M$ is an odd-cycle, however, we have this property. This is equivalent to the statement that cones over odd cycles are 4-chromatic. This was proved by Stiebitz [17] using topological methods. A combinatorial proof is given by Tardif [18] based on the work of El-Zahar and Sauer [3].

**Theorem 4.2** (Stiebitz [17])**.** *Cones over odd cycles are 4-chromatic.* $\qquad\square$

From this we have thus that if $M$ is an odd cycle (say of length $\geq 2q + 5$), then in any 3-coloring of $\mu^q(M)$, 3 colors appear at active vertices (and the third color can be required to appear at only one active vertex of our choice). And it is not hard to check that if any edge from $\mu^q(M)$ is deleted, we can 3-color the result so that at most 2 colors appear at active vertices. These properties imply that if we construct a graph $Y$ from $\mu^q(M)$ by matching the set of active vertices of $\mu^q(M)$ to an (independent) set of equal cardinality, the graph $Y$ will have the same properties under 3-colorings with respect to this independent set as does $U(3)$ (enumerated in Lemma 2.5). Thus if we join in a complete bipartite graph

| $k$ / $\ell$ | 4 | 5 | 6 | 7 | 8 | $\cdots$ |
|---|---|---|---|---|---|---|
| 3 | $\geq \frac{1}{16}$ | $\geq \frac{4}{31}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\cdots$ |
| 5 | $\geq \frac{1}{36}$ | $\geq \frac{3}{35}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\cdots$ |
| 7 | $\geq \frac{1}{64}$ | ? | ? | ? | ? | $\cdots$ |
| 9 | $\geq \frac{1}{100}$ | ? | ? | ? | ? | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |

Table 1: Bounds and exact values for the constants $c_{\ell,k}$. We know nothing for $\ell \geq 7, k \geq 5$, except for the upper bound of $\frac{1}{4}$ from the Erdős-Stone theorem.

these independent sets from two copies of $Y$, the result is a 4-critical graph. It is not hard to check (using Lemma 4.1) that it has odd-girth $2q + 5$. The color classes of the bipartite graph each have size $\frac{1}{2q+4}n$. Thus the result has $> \frac{1}{(\ell+1)^2}n^2$ edges ($\ell$ is 2 less than odd-girth). This proves Theorem 1.5.

## 5   Further Questions

There are many natural questions that follow from what we have presented here. Probably the most immediate is whether our constants $c_{\ell,k}$ are optimal. Our current knowledge on the constants $c_{\ell,k}$, coming from the lower bounds proved in this paper and the trivial $\frac{1}{4}$ upper bound from the Erdős-Stone theorem, are summarized in Table 1. (The bound $c_4 \geq \frac{1}{16}$ comes from Toft's graph and so is not new). The big embarrassment is that we have no positive lower bounds for the constants $c_{\ell,k}$ for any pairs $(\ell, k)$ where $\ell \geq 7$ and $k \geq 5$ (*i.e.*, we do not know that there are families of quadratic density for these parameters). For general critical graphs, on the other hand, $k = 4$ (the Toft graph) was the hardest case; everything else follows from it, since joining a new vertex to all previous vertices has the effect of incrementing the parameter $k$. Similarly, recall from the discussion in Section 1 that for triangle-free graphs ($\ell = 3$), there is a recursive argument using Mycielski's construction to show that all the constants $c_k$ ($k \geq 4$) are positive once it is known (from Toft's graph) that $c_4 > 0$—the significance of Theorem 1.3 is, in the first place, that they are bounded below by a constant (*e.g.*, $\frac{1}{16}$ for $k \geq 4$). Mycielski's construction gives graphs with odd-girth 5, however, so for $\ell = 5$ we know of no such easy argument even just to show that constants $c_{5,k}$ are all positive—we know this just by Theorem 1.4, which shows as well that they are bounded below by a constant—and for $\ell \geq 7$ (and $k > 4$) we have not succeeded at showing this at all.

**Conjecture 5.1.** *There are quadratic-density families of $k$-critical graphs of*

*odd-girth* $\geq \ell$ *for all* $k \geq 4$, $\ell \geq 3$. *In other words, we have for these* $k, \ell$ *that* $c_{\ell,k} > 0$.

It would be particularly nice to have an elegant recursive proof of this conjecture: for example, an operation which, given a (dense) $k$-critical graph of odd-girth $\ell$, produces in some reasonable way a (dense) $(k+1)$-critical graph of odd-girth $\ell$. We expect the following to be true as well:

**Conjecture 5.2.** *For each* $\ell \geq 3$, *the constants* $c_{\ell,k}$ ($k \geq 4$) *are bounded below by a constant. In particular, for each* $\ell$ *there is a quadratic-family of critical graphs of odd-girth* $\ell$ *of arbitrarily large chromatic number.*

Constructions of graphs with large chromatic number and high odd girth are available (*e.g.* [13], [12]), but, typically, they fail to give critical graphs and avoid even short cycles as well as odd ones (precluding the possibility of quadratic edge density). Two constructions that are exceptions are worth mentioning. Though they do not have quadratic edge-density, Kneser graphs give classical examples of highly-chromatic graphs with arbitrary odd-girth (while containing cycles of the even lengths), and their subgraphs known as Stable Kneser graphs found by Schrijver [16] are vertex-critical (weaker than the edge-critical requirement considered in this paper). Their 'continuous' counterparts are the Borsuk graphs, whose finite induced subgraphs have quadratic edge density (the constant depending on $k, \ell$), large chromatic number, and odd-girth (see [14] for related results on Borsuk graphs). It seems we know nothing about critical subgraphs of the Borsuk graphs, however.

As discussed earlier, the families of triangle-free and pentagon-and-triangle-free graphs seem to be rather unusual in that we have succeeded at determining the maximum asymptotic edge-density in $k$-critical members. For these families, however, this determination did not require any sophisticated upper-bound result, since it turns out that the maximum edge density is asymptotically unchanged by the $k$-criticality restriction. In fact, it seems that there is a dearth of upper-bound results about critical graphs—perhaps all known upper bounds come from trivial observations like the fact that a $k$-critical connected graph cannot contain $K_k$ as proper subgraph, which allows an application of Turán's theorem. (For some recent results and background in the area of the density of general critical graphs, see for example [9].)

Because of the added restriction, triangle-free graphs may be a good place to attempt upper bound results on critical graphs. For example, can we show some nontrivial upper bound on the constant $c_4$? With its strong requirements on the odd-girth, the following question may be a good testing ground for attempts at upper bounds.

**Question 5.3.** *What is the behavior of the constants* $c_{\ell,k}$ *as* $\ell$ *grows large? Is it true that* $c_{\ell,4} \to 0$ *as* $\ell \to \infty$? *Or is there perhaps some positive lower bound for all the constants* $c_{\ell,k}$?

One can also return to the question of the size of the minimum degree in highly-chromatic triangle-free graphs. Since the Erdős-Simonovits-Hajnal construction [5] is far from being critical, we ask:

15

**Question 5.4.** *How large a minimum degree forces the chromatic number of triangle-free (resp. odd-girth $> \ell$) critical graphs to be bounded?*

Unlike the case where we do not require criticality, we have no linear lower bound. Thomassen has shown that we cannot have a linear lower bound on the minimum degree if we avoid higher order odd-cycles, even without requiring criticality [20].

# References

[1] S. Brandt. On the structure of dense triangle-free graphs. *Combinatorica*, 20:237–245, 1999.

[2] S. Brandt and S. Thomassé. Dense triangle-free graphs are 4-colorable: A solution to the Erdős-Simonovits problem. `http://www.lirmm.fr/~thomasse/liste/vega11.pdf`.

[3] M. El-Zahar and N. Sauer. The chromatic number of the product of two 4-chromatic graphs is 4. *Combinatorica*, 5:121–126, 1985.

[4] P. Erdős, D. J. Kleitman, and B. L. Rothschild. Asymptotic enumeration of $K_n$-free graphs. In *Colloquio Internazionale sulle Teorie Combinatorie (Rome, 1973), Tomo II*, pages 19–27. Atti dei Convegni Lincei, No. 17. Accad. Naz. Lincei, Rome, 1976.

[5] P. Erdős and M. Simonovits. On a valence problem in extremal graph theory. *Discrete Mathematics*, 5:323–334, 1973.

[6] P. Erdős and A. H. Stone. On the structure of linear graphs. *Bulletin of the American Mathematical Society*, 52:1087–1091, 1946.

[7] A. Gyárfás. Personal communication, 2005.

[8] R. Häggkvist. Odd cycles of specified length in nonbipartite graphs. In *Graph Theory*, pages 89–99. Annals of Discrete Mathematics, Vol. 13. North-Holland, Amsterdam-New York, 1982.

[9] T. R. Jensen. Dense critical and vertex-critical graphs. *Discrete Mathematics*, 258:63–84, 2002.

[10] G. Jin. Triangle-free four-chromatic graphs. *Discrete Mathematics*, 145:151–170, 1995.

[11] A. V. Kostochka and M. Stiebnitz. On the number of edges in colour-critical graphs and hypergraphs. *Combinatorica*, 20:521–530, 2000.

[12] I. Kříž. A hypergraph-free construction of highly chromatic graphs without short cycles. *Combinatorica*, 9:227–229, 1989.

[13] L. Lovász. On chromatic number of graphs and set-systems. *Acta Math. Hungar.*, 19:59–67, 1968.

[14] L. Lovász. Self-dual polytopes and the chromatic number of distance graphs on the sphere. *Acta Scientiarum Mathematicarum*, 45:317–323, 1983.

[15] J. Mycielski. Sur le coloriage des graphes. *Colloq. Math.*, 3:161–162, 1955.

[16] A. Schrijver. Vertex-critical subgraphs of kneser graphs. *Nieuw Arch. Wiskd. III. Ser.*, 26:454–461, 1978.

[17] M. Stiebitz. *Beiträge zur Theorie der färbungskritschen Graphen.* PhD thesis, Technical University Ilmenaur, 1985.

[18] C. Tardif. Fractional chromatic numbers of cones over graphs. *Journal of Graph Theory*, 38:87–94, 2001.

[19] C. Thomassen. On the chromatic number of triangle-free graphs of large minimum degree. *Combinatorica*, 22:591–596, 2002.

[20] C. Thomassen. On the chromatic number of pentagon-free graphs of large minimum degree. *Combinatorica*, 27:241–243, 2007.

[21] B. Toft. On the maximal number of edges of critical $k$-chromatic graphs. *Studia Sci. Math. Hungar.*, 5:461–470, 1970.

[22] A. Zykov. On some properties of linear complexes (in russian). *Matem. Sbornik*, 24:163–187, 1949.